Statistical Physics (PH312)

HW #3, Fall 2018

due : November 15, 2018

Note that you SHOULD show the details of your work.

1. Consider an electron subject to a magnetic field $\vec{B}$, which is described by the Hamiltonian $\mathcal{H}$ as

$$\mathcal{H} = -\mu_B(\vec{\sigma} \cdot \vec{B})$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli spin matrices and μ_B is the Bohr magneton.
(a) Find the density matrix ρ and the expectation value of σ_z in the canonical ensemble if $\vec{B}$ is applied in the z direction.
(b) Repeat the same calculation to find ρ and $\langle \sigma_x \rangle$ assuming that $\vec{B}$ is along the x direction.
(c) What is the average energy in each of the above cases?

2. (a) In the class, we have used the symmetrized state vectors for an quantum ideal gas of N identical particles to evaluate the density matrix and the canonical partition function. Use the unsymmetrized state vectors instead of the symmetrized ones and following the same procedures as done in the class, calculate the density matrix $\langle x_1', x_2', \cdots, x_N' | \rho | x_1, x_2, \cdots, x_N \rangle$ and the partition function. From this, you can confirm yourself that without the symmetrization, neither the Gibbs' factor nor statistical interaction between particles is derived.
(b) Find the equations of state for an ideal Bose gas and an ideal Fermi gas in the high temperature and low density limit, up to the first correction due to quantum effects. This is what we have already learned in the class.

3. Consider a system where the occupation number $n_\mathbf{k}$ of an one-particle state $\mathbf{k}$ is restricted to the values $0, 1, \cdots, \ell$.
(a) Show that the average occupation number is given by

$$\langle n_\mathbf{k} \rangle = \frac{1}{z^{-1} e^{\beta\epsilon(\mathbf{k})} - 1} - \frac{\ell + 1}{(z^{-1} e^{\beta\epsilon(\mathbf{k})})^{\ell+1} - 1}.$$

(b) Check that while $\ell = 1$ leads to the Fermi-Dirac statistics, $\ell \to \infty$ leads to the Bose-Einstein statistics.

4. Consider a system of an ideal Fermi gas where the density of states is given as

$$g(\epsilon) = \begin{cases} D & \text{for } \epsilon > 0\,, \\ 0 & \text{for } \epsilon < 0\,, \end{cases}$$

where D is a constant.
(a) Calculate the Fermi energy for this system.
(b) Determine the condition for the system being highly degenerate.
(c) Show that the heat capacity is proportional to T for the highly degenerate case.