Nonequilibrium Statistical Mechanics

HW #2, Spring 2020

due : June 19, 2020

Note that you SHOULD show the details of your work.

1. (a) The characteristic function of multivariate Gaussian distribution has the form

$$G(k_1, k_2, \ldots, k_n) = e^{i\sum_i k_i\mu_i - \frac{1}{2}\sum_{i,j} k_i k_j \sigma_{ij}}.$$

By explicit calculation, find the probability distribution and show that $\langle X_i X_j \rangle_c = \sigma_{ij}$. Also show that all higher moments can be expressed in terms of products of first two moments and that all cumulants higher than second order vanish.

(b) Show that if $X(t)$ is a Gaussian process, any linear transformation of it

$$Y(t) = \int_a^b c(t, t')X(t')dt'$$

is also Gaussian.

2. Consider the Caldeira-Leggett model where a particle of mass m, position x, and conjugate momentum p, under an external potential $U(x)$ is coupled with a bath of N independent harmonic oscillator of mass m_j, position x_j and momenta p_j. Then the Hamiltonian of the system reads as

$$H = \frac{p^2}{2m} + U(x) + \frac{1}{2}\sum_{j=1}^{N}\left[\frac{p_j^2}{2m_j} + m_j\omega_j^2\left(x_j - \frac{c_j}{m_j\omega_j^2}x\right)^2\right].$$

Show that there is no correlation between different modes and the generalized fluctuation-dissipation theorem can be obtained as

$$\langle f(t)f(t')\rangle_0 = mk_BT\gamma(t - t')$$

where the memory kernel is given as $\gamma(t) = (1/m)\sum_j c_j/(m_j\omega_j^2)\cos\omega_j t$. Note that $\langle\ldots\rangle_0$ denotes ensemble average over initial equilibrium distribution.

3. Eigenfunction expansion method: consider the Langevin equation for the velocity of Brownian particle in the absence of external driving force (Ornstein-Uhlenbeck process):

$$\frac{dv}{dt} = -\gamma v + \xi(t).$$

(a) Obtain the Fokker-Planck equation for the probability to find the particle with velocity v at time t, $P(v, t)$ and find the Fokker-Planck operator $\mathcal{L}_{FP}$, explicitly. Show that

$$\Phi(v) = \frac{\gamma v^2}{2D},\ \mathcal{F}(v) = -\gamma v$$

(b) Using the following operator

$$\begin{aligned} b &= \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\xi} + \xi\right) \\ b^\dagger &= \frac{1}{\sqrt{2}}\left(-\frac{\partial}{\partial\xi} + \xi\right) \end{aligned}$$

where $\xi = v\sqrt{\gamma/2D}$, show that the FPE can be transformed into eigenvalue problem with Hermitian operator $\mathcal{H} = \gamma b^\dagger b$. Using the eigenfunctions that can be expressed in terms of Hermite polynomials, find the transition probability $P(v, t|v_0, 0)$ when the initial probability distribution is $P(v, 0) = \delta(v - v_0)$.