Nonequilibrium Statistical Mechanics

HW #2, Spring 2019

due : April 30, 2019

Note that you SHOULD show the details of your work.

1. (a) Let X_j be an infinite set of independent random variables with identical distribution $P(x)$ and characteristic function $G(k)$. Let r be a random positive integer with distribution p_r and probability generating function $f(z)$ [the generating function can be obtained from the characteristic function more conveniently by setting $s = -ik$, $F(z) = \langle z^X \rangle = \langle e^{-sX} \rangle$, when X only takes positive values]. Show then that the characteristic function of a random variable $Y = \sum_{j=1}^{r} X_j$ is $F(G(k))$.
(b) Consider the Wiener process with the transition probability
$$P(y,t|y',t') = \frac{1}{\sqrt{2\pi(t-t')}} \exp\left[-\frac{(y-y')^2}{2(t-t')}\right].$$
Prove for the Wiener process, when $0 < t_1 < t_2$,
$$\langle y_2 \rangle_{y_1} = y_1, \qquad \langle\langle y_2^2 \rangle\rangle_{y_1} = t_2 - t_1$$
$$\langle y_1 \rangle_{y_2} = \frac{t_1}{t_2} y_2, \qquad \langle\langle y_1^2 \rangle\rangle_{y_2} = \frac{t_1}{t_2}(t_2 - t_1).$$
where y_1, y_2 stand for $Y(t_1), Y(t_2)$. Here $\langle \ldots \rangle_z$ indicates a conditional average with constant z and $\langle\langle \ldots \rangle\rangle$ denotes a cumulant.
Also show that for the Wiener process,
$$\langle y^n(t) \rangle = \begin{cases} (n-1)!!\, t^{n/2}, & \text{even } n \\ 0, & \text{odd } n \end{cases}$$
$$\langle (\Delta y)^n \rangle = \begin{cases} (n-1)!!\, (\Delta t)^{n/2}, & \text{even } n \\ 0, & \text{odd } n \end{cases}$$
$$\langle Y(t_1) Y(t_2) \rangle = \min(t_1, t_2)$$
(c) Show that if $X(t)$ is a Gaussian random variables, then
$$\langle x^*(t)x(t)x^*(0)x(0) \rangle = |\langle x^*(0)x(0) \rangle|^2 + |\langle x^*(t)x(0) \rangle|^2.$$
(d) Take a sequence of variables $X_j (j = 1, 2, \ldots, r)$ with distributions $P_j(x) = f(x - j)$ with fixed f. Show that the central limit property does not apply but that the variable Z defined by
$$\sum_{j=1}^{r} X_j = \frac{1}{2} r(r+1) + Z$$
does tend to a Gaussian. How can this be seen *a priori*?

2. Kramers-Moyal backward expansion: from the following Chapman-Kolmogorov equation for the transition probability
$$P(q,t|q',t') = \int dq'' P(q,t|q'',t'+\tau)\, P(q'',t'+\tau|q',t')$$
where $t \geq t' + \tau \geq t'$. Derive the Kramers-Moyal backward expansion where we differentiate P with respect to q' and t', i.e., with respect to the value of the stochastic variable at earlier time $t' < t$:
$$\frac{\partial}{\partial t} P(q,t|q',t') = -\mathcal{L}_{KM}^{\dagger}(q',t') P(q,t|q',t')$$
with
$$\mathcal{L}_{KM}^{\dagger}(q',t') = \sum_{n=1}^{\infty} \mathcal{D}^{(n)}(q',t') \left(\frac{\partial}{\partial q'}\right)^n.$$

3. Stochastic calculus: for general α, evaluate the averages of the following stochastic integrals,

$$\int_0^t dW(s)W(s)s, \quad \int_0^t dW(s)W^3(s)e^{-\lambda s}, \quad \int_0^t dW(s)W^{2k+1}(s)$$

4. Suppose that we have a random variable satisfying the following stochastic differential equation,

$$dx = -\beta x dt + \sqrt{2\beta(a^2 - x^2)}dW(t)$$

where $x \in [-a, a]$.
(a) Assuming the Ito interpretation ($\alpha = 0$), find the corresponding Fokker-Planck equation.
(b) Find the normalized steady state probability $P_s(x)$.
(c) Consider the general (forward) Fokker-Planck equation written as

$$\frac{\partial}{\partial t}P(x,t) = \mathcal{L}_{FP}P(x,t)$$

with the Fokker-Planck operator $\mathcal{L}_{FP}$. What is the explicit expression of $\mathcal{L}_{FP}$ with the position-dependent drift and diffusion coefficient, $A(x)$ and $B(x)$? Defining $P(x,t) = P_s(x)Q(x,t)$, show that Q satisfies

$$\frac{\partial}{\partial t}Q(x,t) = \mathcal{L}^{\dagger}_{FP}Q(x,t)$$

where $\mathcal{L}^{\dagger}_{FP}$ is the backward Fokker-Planck operator. Now consider the eigenfunctions $P_n(x)$ and $Q_n(x)$ which satisfy

$$\mathcal{L}_{FP}P_n(x) = -\lambda_n P_n(x), \quad \mathcal{L}^{\dagger}_{FP}Q_n(x) = -\lambda_n Q_n(x)$$

For $\mathcal{L}_{FP}$ given as (a), solve for $P_n(x)$ and $Q_n(x)$. Hint: use a Chebyshev polynomials.
(d) Obtain an expression for $\langle x^3(t)x^3(0)\rangle$ when $x(0) = x_0$ is distributed according to $P_s(x_0)$.