Statistical Physics (PH312)

HW #4, Fall 2019

due : Nov 29, 2019

Note that you SHOULD show the details of your work.

1. Show that, if the occupation number n_ϵ of an energy level ϵ is restricted to the values $0, 1, \cdots, l$, then the mean occupation number of that level is given by

$$\langle n_\epsilon \rangle = \frac{1}{z^{-1}e^{\beta\epsilon} - 1} - \frac{l+1}{(z^{-1}e^{\beta\epsilon})^{l+1} - 1}$$

Also, Check that while $l = 1$ leads to $\langle n_\epsilon \rangle_{F.D}$, $l \to \infty$ leads to $\langle n_\epsilon \rangle_{B.E}$.

2. Consider a system with total particle number N. Each particles can occupy a state from set of states $\{\varepsilon\}$. Expression of the entropy depends on type of particles as following

$$S = k_B \sum_\varepsilon [< n_\varepsilon > ln < n_\varepsilon + 1 > - < n_\varepsilon > ln < n_\varepsilon >] \quad (for\ \ bosonic\ \ particles)$$

$$S = k_B \sum_\varepsilon [- < 1 - n_\varepsilon > ln < 1 - n_\varepsilon > - < n_\varepsilon > ln < n_\varepsilon >] \quad (for\ \ fermionic\ \ particles)$$

Dervie those two formulas from general formula $S = -k_B \sum_\varepsilon [\sum_n p_\varepsilon(n)\, ln\, p_\varepsilon(n)]$.

3. Consider an one-dimensional quantum mechanical harmonic oscillator which is in equilibrium with a heat reservoir at temperature T. The Hamiltonian of the oscillator is given by

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2. \tag{1}$$

Then energy eigenvalues E_n and corresponding wave functions Ψ_n are determined as

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega, \quad \Psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{H_n(s)}{\sqrt{2^n n!}} \exp\left[-\frac{1}{2}s^2\right]. \tag{2}$$

where H_n is the nth Hermite polynomial, and $s = (m\omega/\hbar)^{1/2}x$.

(a) Find a coordinate representation of density operator $\hat{\rho}$ of the system in terms of H_n's.

(b) To develop the result of (a), we can use the integral representation of H_n

$$H_n(x) = \frac{e^{x^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} (-2iu)^n e^{-u^2+2ixu} du. \tag{3}$$

Substituting H_n in the result of (a) to above integral representation, show that the diagonal elements of $\hat{\rho}$ in the coordinate representation is given by

$$\rho(x) \equiv \langle x|\hat{\rho}|x\rangle = \left[\frac{m\omega}{\pi\hbar}\tanh\left(\frac{1}{2}\beta\hbar\omega\right)\right]^{1/2} \exp\left[-\frac{m\omega}{\hbar}\tanh\left(\frac{1}{2}\beta\hbar\omega\right)x^2\right] \tag{4}$$

(c) Calculate the ensemble average $\langle x^2 \rangle$ and $\langle p^2 \rangle$ by using the density operator. Do those agree with the result of the classical case?