Statistical Physics (PH312)

HW #2, Fall 2019

due : Oct 15, 2019

Note that you SHOULD show the details of your work.

1. Consider an one-dimensional classical harmonic oscillator which is in equilibrium with a heat reservoir at temperature T. The Hamiltonian of the oscillator is given by

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$

(a) Calculate the partition function Z_{cl}.
(b) Calculate the ensemble average $\langle p^2 \rangle$, $\langle x^2 \rangle$, $\langle E \rangle$ and verify the equipartition theorem.
(c) Now consider a quantum mechanical harmonic oscillator the energy eigenvalue of which is then given as

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega.$$

Compute the partition function Z_{QM}.
(d) By using Z_{QM}, compute the ensemble average $\langle E \rangle$ and verify the result of (b) by letting $T \to \infty$.

2. Consider a crystal of non-interacting N atoms having a spin of 1/2, i.e., $s^z = \pm\frac{1}{2}$. The Hamiltonian of such a system in a magnetic field $\mathbf{H} = H\hat{z}$ along the $\hat{z}$ axis is

$$\mathcal{H} = -\sum_{i=1}^{N} \boldsymbol{\mu}_i \cdot \mathbf{H} = -\sum_{i=1}^{N} g\mu_B H s_i^z$$

where the magnetic moment of the i-th atom is $\boldsymbol{\mu}_i = g\mu_B \mathbf{s}_i$, where g is the Lande g-factor, and $\mu_B = e\hbar/2mc$ is the Bohr magneton. Suppose that the system is in thermal equilibrium at temperature T.
(a) Show that the partition function is $Z = (2\cosh\eta)^N$ where $\eta = g\mu_B H/2k_B T$.
(b) Find an expression for the entropy S (you need only consider the contributions from the spin states). Evaluate S in the strong field ($\eta \gg 1$) and weak field ($\eta \ll 1$) limits.
(c) An important process for cooling substances below 1 K is adiabatic demagnetization. In this process the magnetic field on the sample is increased from 0 to H_0 while the sample is in contact with a heat bath at temperature T_0. Then the sample is thermally isolated and the magnetic field is reduced to $H_1 < H_0$. What is the final temperature of the sample?
(d) The magnetization M_z and susceptibility χ are defined by $M_z = \left\langle \sum_{i=1}^{N} (\boldsymbol{\mu}_i)_z \right\rangle$ and $\chi = \frac{\partial M_z}{\partial H}$, respectively. Find expressions for M_z and χ, and evaluate these expressions in the weak field limit.

3. Consider a two-dimensional ultrarelativistic gas where N non-interacting massless particles in thermal equilibrium with a heat reservoir at temperature T and are confined in area A. The Hamiltonian of such a gas is given as $H(\vec{q}, \vec{p}) = \sum_{i=1}^{N} |\vec{p}_i| c$ where c is speed of light.
(a) For general case of canonical ensemble, derive that $F = -k_B T \ln Z$ where F is the Helmholtz free energy and Z is the partition function.
(b) Calculate the free energy F (use the Stirling approximation).
(c) Calculate the entropy S, internal energy U, pressure P and chemical potential μ.